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The stability of the decaying flow in a suddenly blocked channel

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The stability of the decaying laminar flow in a suddenly blocked channel is investigated. The partial differential system governing the stability of the flow is solved using a WKB type of approach. It is shown that the first term of the WKB expansion of the disturbance velocity field is just that obtained by a quasi-steady approach. The flow is found to be unstable at quite small Reynolds numbers. This instability is associated with the inflexional nature of the velocity profiles of the decaying flow.

1. Introduction

The laminar decay of a fully developed flow in a pipe or channel which is suddenly blocked by, for example, the rapid closure of a valve was the subject of a recent paper (Weinbaum & Parker 1975). That paper, hereafter referred to as I, was motivated by an interest in the stability of decelerating flows aroused by the observation of disturbances which appear during the deceleration phase of aortic blood flow (see for example Nerem & Seed 1972).

The basic flow discussed in I develops on two separate time scales. The first is a short time scale h/c, where h is the channel half-height and c is the speed of sound in the fluid, which is characteristic of the time of passage of a pressure wave.

The second is a longer time scale h^2/ν, where ν is the kinematic viscosity. This time is characteristic of the rate of diffusion of momentum.

Immediately after the channel is blocked, the flow adjusts to the new boundary condition of zero net flow by means of a pressure wave. This adjustment occurs on the short time scale. In the incompressible limit, it is simply an impulsive uniform deceleration of the entire velocity profile by an amount equal to the average velocity of the initial, undisturbed profile.

The flow immediately after the passage of the pressure wave satisfies the condition of zero net flow but results in a slip velocity at the wall. In order to satisfy the no-slip condition at the wall a boundary layer develops on the longer diffusion time scale. For full details of the calculation of the flow during this time the reader is referred to I. We note simply that an approximate solution was obtained using a Pohlhausen technique.

In this paper we investigate the linear stability of the flow described above using a quasi-steady approach. Thus we fix upon a particular profile at some time and consider the stability of this profile as if it were a steady flow. This is justifiable if there exists a fast time scale on which a disturbance can grow before the basic flow changes significantly. In such a case, the quasi-steady approach can be shown to represent the first term of an asymptotic expansion of the WKB type. We shall see that this type of approach can be justified in the problem considered in this paper. The WKB method has previously been used in unsteady hydrodynamic stability theory by Rosenblat & Herbert (1970) and Seminara & Hall (1975) and was first suggested by Benney & Rosenblat (1964). The method has also been used for spatially slowly varying flows by Bouthier (1973), Gaster (1974), Drazin (1974) and Eagles & Weissmann (1975).

The procedure adopted in this paper is as follows. In §2 we formulate the differential system governing the stability of the flow discussed in I. In §3 we obtain an asymptotic solution to this system in the limit of large Reynolds numbers. In §4 we discuss the results obtained in §3 and show how they can be related to the results of an inviscid analysis.

2. Formulation of the problem

Suppose that for times $\bar{t} < 0$ we have fully developed flow in a two-dimensional channel defined by $-\infty \leqslant X \leqslant \infty$, $-h \leqslant Y \leqslant h$. If u is the velocity along the channel we have

$$u = U_0(1 - Y^2/h^2). \tag{2.1}$$

The laminar decay of this profile when a valve is closed at time $\bar{t} = 0$ has been discussed in I. We shall concern ourselves only with the stability of this flow when it is developing on the diffusion time scale $t_D = h^2/\nu$. We first define the following dimensionless variables:

$$x = X/h, \quad y = Y/h, \quad t = \bar{t}/t_D. \qquad (2.2a, b, c)$$

We further restrict our attention to a region far enough away from the valve so that the basic flow can be considered unidirectional. Calculations in I for the flow near the valve showed that this condition is satisfied at distances greater than about two channel heights from the valve. In this region, and on the diffusion time scale, the non-dimensional velocity of the flow in the x direction is the function $u(y,t)$ evaluated in I. It should also be pointed out that the experimental observations made on a pipe showed that the instability did not arise near the valve.

It is instructive at this stage to point out the important feature of $u(y,t)$. Some typical profiles of $u(y,t)$ for different values of t are shown in figure 1. For each value of t there exists an inflexion point in the interval $0 \leqslant y \leqslant 1$. This immediately alerts us to the occurrence of inflexion-point instabilities, since we know from steady inviscid stability theory that the existence of an inflexion point is a necessary condition for instability. However it is dangerous to infer that such instabilities necessarily occur for a time-dependent flow. For example the results of von Kerczek & Davis (1974) show that a Stokes layer, which can have one or more inflexion points at different times, is stable at least up to Reynolds numbers of about 750. Von Kerczek & Davis argue that the profiles do not live long enough for any instability associated with the inflexion point to grow.

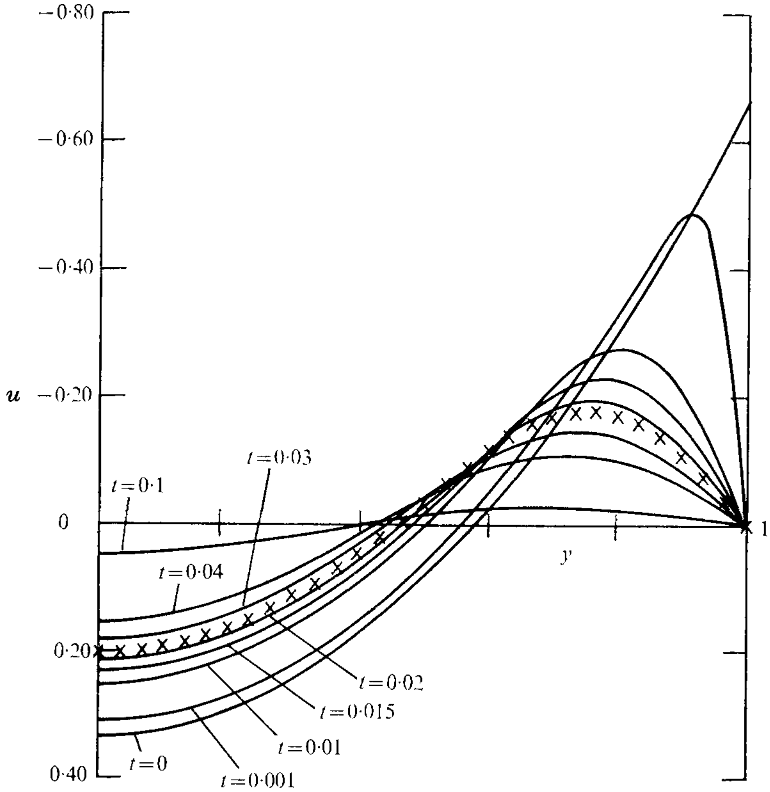

FIGURE 1. The basic flow velocity profiles at different times. × × ×, the most unstable profile, at $t = 0{\cdot}023$.

The crucial question in our problem is whether there exists another (fast) time scale on which an instability can grow before the profiles change significantly. With this in mind we consider the convective time scale $t_c = h/U_0$, and thus define a dimensionless time scaled on t_c by writing

$$T = \bar{t}/t_c.$$

If this time scale is to be short compared with the diffusion time scale we require that

$$t_D/t_c = U_0 h/\nu = R \gg 1.$$

Thus for large values of the Reynolds number R we expect inflexional instabilities to occur. Suppose that we perturb (two-dimensionally) the flow $(u, 0)$ such that the stream function $\psi(x, y, t, T)$ of the disturbance is of the form

$$\psi(x, y, t, T) = \epsilon\Psi(y, t, T) \exp\{i\alpha x\} + \text{complex conjugate}, \tag{2.3}$$

so that the disturbance is periodic in the flow direction. If the amplitude ϵ of the disturbance is allowed to tend to zero it is an easy matter to show that, neglecting terms of order ϵ^2, Ψ is determined by

$$\left\{u - \frac{i}{\alpha R}\frac{\partial}{\partial t} - \frac{i}{\alpha}\frac{\partial}{\partial T}\right\}\left\{\frac{\partial^2}{\partial y^2} - \alpha^2\right\}\Psi - \frac{\partial^2 u}{\partial y^2}\Psi + \frac{i}{\alpha R}\left\{\frac{\partial^2}{\partial y^2} - \alpha^2\right\}^2\Psi = 0. \tag{2.4}$$

We also require that the disturbance velocity is zero at the wall. Thus Ψ must satisfy the following boundary conditions:

$$\Psi = \partial\Psi/\partial y = 0, \quad y = \pm 1. \tag{2.5}$$

In the next section we obtain an asymptotic solution of (2.4) and (2.5) in the limit $R \to \infty$.

3. The limit $R \to \infty$

We have seen in § 2 that when $R \to \infty$ there exists a fast convective time scale $t_c = hU_0^{-1}$ and a slow diffusion time scale $h^2\nu^{-1}$. In such a situation we can look for a solution of the WKB type. Thus, following for example Seminara & Hall (1975), we drop the explicit dependence of Ψ on the fast time variable T and look for a solution of the form

$$\Psi = \exp\left\{-i\alpha R\int_0^t c(\tau)\,d\tau\right\}\left\{\psi_0(y,t)+\frac{1}{R}\psi_1(y,t)+\ldots\right\}. \tag{3.1}$$

However, it is clear that the exponential term above represents an implicit dependence of Ψ on the fast time variable T. At any given value of the slow time variable t the flow is stable or unstable depending on whether the imaginary part of $c(t)$ is negative or positive respectively. It is not necessary at this stage to expand c in powers of R^{-1} since, as for the WKB solution expansion of an ordinary differential system, the resulting slow dependence of Ψ on t can be absorbed into the functions ψ_0, ψ_1, etc. Suppose that we fix α and R; then, for any given value of t, we must solve for the corresponding values of c, ψ_0, ψ_1, etc. If we substitute from (3.1) into (2.4) and (2.5) and equate terms of order R^0 we find that ψ_0 can be written in the form

$$\psi_0 = A(t)\,\Psi_0(y,t), \tag{3.2}$$

where $A(t)$ is a function of t to be determined and Ψ_0 satisfies

$$\left.\begin{aligned} \{u-c\}N\Psi_0-\frac{\partial^2 u}{\partial y^2}\Psi_0+\frac{i}{\alpha R}N^2\Psi_0 &= 0,\\ \Psi_0 = \partial\Psi_0/\partial y = 0, \quad y &= \pm 1, \end{aligned}\right\} \tag{3.3}$$

where

$$N \equiv d^2/dy^2-\alpha^2. \tag{3.4}$$

This is an ordinary differential system, parametrically dependent on t. In fact, the above differential system is identical to that which we would have obtained by making a quasi-steady approximation initially and solving the Orr–Sommerfeld equation at any value of t with the basic velocity profile $u(y,t)$ as if it were steady. Thus we see that the first term of a WKB type of expansion is just the disturbance velocity obtained by a quasi-steady approach.

We note at this stage that in (3.3) we have retained the viscous term $R^{-1}N^2\Psi_0$ even though it is formally of lower order in the Reynolds number. This term is required to smooth out any singularities of the inviscid operator in any critical layers and to enable the no-slip condition to be satisfied at the wall. A similar term is retained at first order in the WKB-type solutions of Bouthier (1973) and Gaster (1974), who considered the stability of a boundary layer on a flat plate.

At any value of t, (3.3) determines an eigenrelation of the form

$$c(t) = f(\alpha, R, t). \tag{3.5}$$

At this stage, following the suggestion of a referee, we could proceed by perturbing about the inviscid limit and expanding $c(t)$ and $\Psi_0(y,t)$ in inverse powers of R. However such an approach would enable us to find only the upper branch of the neutral curve. Moreover, since the critical value of R is in fact of order 10^2, the numerical solution of (3.3) presents no great difficulties. Therefore we feel that, having solved (3.3) numerically, there is little to be gained by such a procedure. At order R^{-1} we find that ψ_1 is determined by

$$\left.\begin{aligned} \{u-c\}N\psi_1 - \frac{\partial^2 u}{\partial y^2}\psi_1 + \frac{i}{\alpha R}N^2\psi_1 &= \frac{-1}{i\alpha}\frac{dA}{dt}N\psi_0 - \frac{N}{i\alpha}\frac{\partial \Psi_0}{\partial t}A, \\ \psi_1 = \partial\psi_1/\partial y = 0, \quad y &= \pm 1. \end{aligned}\right\} \tag{3.6}$$

Apart from the inhomogeneous terms in (3.6), the differential systems for Ψ_0 and ψ_1 are identical. In such circumstances we can show that (3.6) has a solution only if a certain orthogonality condition is satisfied. This condition is found to take the form

$$A^{-1}\frac{dA}{dt} = -\int_{-1}^{1}\psi_0^+ N\frac{\partial \Psi_0}{dt}dy \Big/ \int_{-1}^{1}\psi_0^+ N\Psi_0 dy, \tag{3.7}$$

where ψ_0^+ is the adjoint function associated with $\Psi_0(y,t)$. Thus we see that the orthogonality condition obtained at order R^{-1} determines the unknown function $A(t)$ obtained at order R^0. Indeed, it is an easy matter to show that the orthogonality condition determined at order R^{-n} determines the unknown function of t obtained at order R^{-n+1}.

4. Results and discussion

Before discussing the numerical work it is necessary to define the term 'growth rate' for an unsteady flow. This point has been discussed in detail by Shen (1961) and Seminara & Hall (1975). If θ is some property of the disturbance such as velocity, kinetic energy, etc., we define the growth rate G by

$$G = +\mathrm{Re}\left\{\frac{\partial\theta/\partial t}{\alpha\theta}\right\}. \tag{4.1}$$

This definition leads to an expression for $G(y,t)$ of the form

$$G = \mathrm{Im}\{c(t) + R^{-1}f(y,t) + O(R^{-2})\}. \tag{4.2}$$

The function $f(y,t)$ depends on which property of the flow we use to define G and in general will be a function of y and t. However, when G is defined in terms of some integrated property of the flow, such as kinetic energy, G is a function only of t.

We can see from (4.2) that as $R \to \infty$ the growth rate differs from its quasi-steady value $\mathrm{Im}\{c(t)\}$ by only an amount of order R^{-1}. Thus as a first approximation we take G to be $\mathrm{Im}\{c(t)\}$. Clearly, better approximations to G can be obtained by solving for the higher-order eigenfunctions ψ_1, ψ_2, etc.

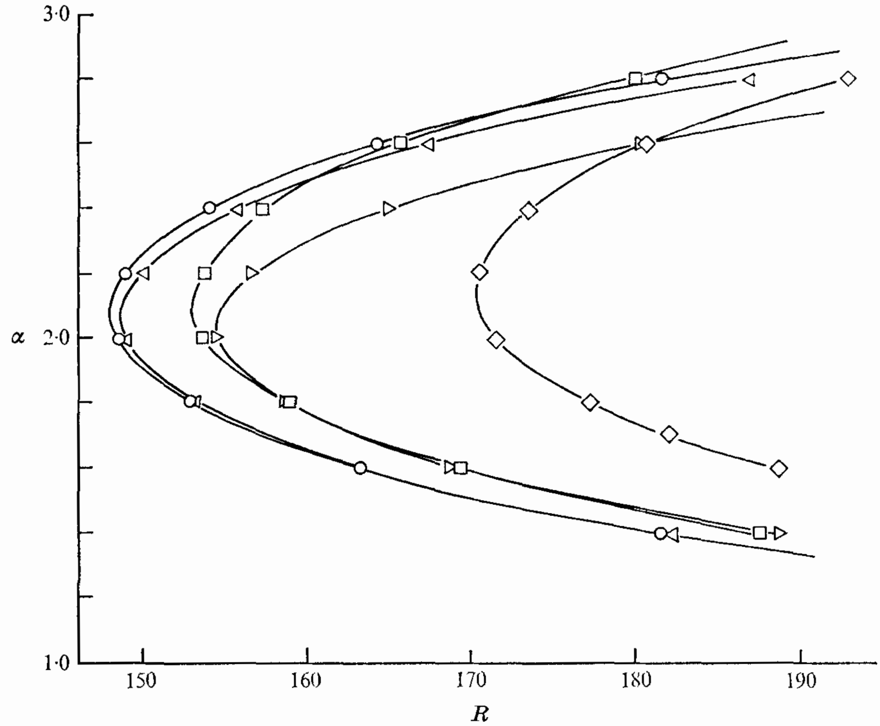

FIGURE 2. The neutral curve at different values of t. ▷, $t = 0{\cdot}015$; ◁, $t = 0{\cdot}02$; ○, $t = 0{\cdot}023$; □, $t = 0{\cdot}03$; ◇, $t = 0{\cdot}04$.

We can then define an instantaneous neutral state by imposing the condition

$$\operatorname{Im}\{c(t)\} = 0. \tag{4.3}$$

For given values of α, R and t we solved (3.3) numerically to determine $c(t)$. This was done using the complete orthonormalization procedure suggested by Davey (1973). Although not strictly necessary for R of order 10^2, this procedure was required for larger values of R. We restricted our attention to even solutions of (3.3).

In figure 2 we show the instantaneous neutral curves for several values of t. As t increases from zero, the minimum value of R on each of these curves, which we denote by $R_{\min}$, first decreases and then increases. The critical time at which $dR_{\min}/dt$ is zero is $t = t^* = 0{\cdot}023$ and is shown in figure 2. The corresponding values of R, c_r and α, which we denote by R_m^*, c_{rm}^* and α_m^* respectively, are

$$R_m^* = 148{\cdot}2, \quad c_{rm}^* = -0{\cdot}046, \quad \alpha_m^* = 2{\cdot}083. \tag{4.4}$$

Thus, when $R = R_m^*$, all disturbances will be instantaneously damped when $t \neq t^*$. When $t = t^*$ all disturbances except the one with $\alpha = \alpha_m^*$ will be damped. Suppose now that $R > R_m^*$. In this case there will be a range of values of t, say $[t_1, t_2]$, such that if t lies outside this range all disturbances will be instantaneously damped. For each value of t inside this range there will be an interval $[\alpha_1(t), \alpha_2(t)]$ such that all disturbances with α in this range instantaneously grow exponentially in time.

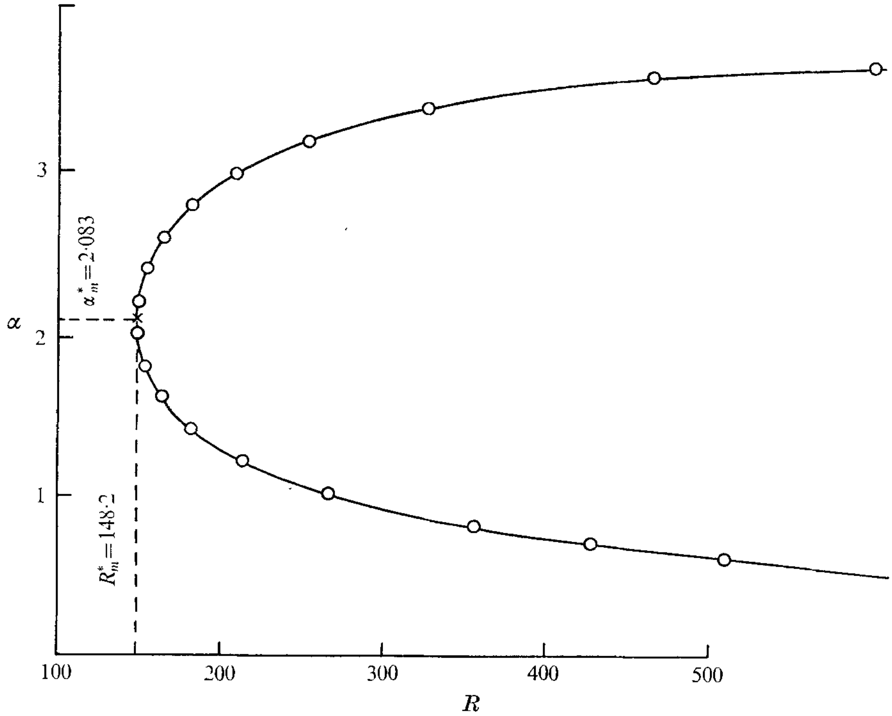

FIGURE 3. The neutral curve for the most unstable profile, i.e. when $t = 0{\cdot}023$.

In figure 3 we show the neutral curve for $t = t^*$ for a large range of values of the Reynolds number. The lower branch of this curve asymptotes to the line $\alpha = 0$ as $R \to \infty$. The upper branch is asymptotic to a line $\alpha = \alpha_c$ as $R \to \infty$. We shall see later that this value of α can be obtained from inviscid stability theory.

In figure 4 (*a*) we have plotted u_I, the velocity of the basic flow at the inflexion point, as a function of t. In figures 4 (*d*), (*b*) and (*c*) we have plotted $R_{\min}$ and the corresponding values of c_r and α (i.e. $c_{r\min}$ and $\alpha_{\min}$ respectively) as functions of t. We note that $\alpha_{\min}$ increases monotonically in time. We can see from figures 4 (*a*) and (*b*) that u_I and c_r have a similar behaviour as functions of t.

In figure 5 we have plotted the wave speed c_r as a function of α for different values of t. Apart from the most dangerous profile at $t = 0{\cdot}023$ we have restricted ourselves to values of α less than 3. For $\alpha > 3$ the upper branch of the neutral curve becomes flatter and finally asymptotes to the line $\alpha = \alpha_c$. Thus the corresponding Reynolds numbers become large and the numerical solution of (3.3) requires a lot of computer time. However, it is likely that the behaviour exhibited by the profile at $t = 0{\cdot}023$ is typical. We see that the wave speed of the most dangerous profile increases monotonically as α increases until it reaches a maximum value. For this profile this maximum value of c_r is quite close to zero. However it is clear from figure 5 that for the other profiles this maximum value will be non-zero. After reaching a maximum the wave speed then decreases monotonically. The last point computed was for $\alpha = 4{\cdot}2$, which corresponds to a

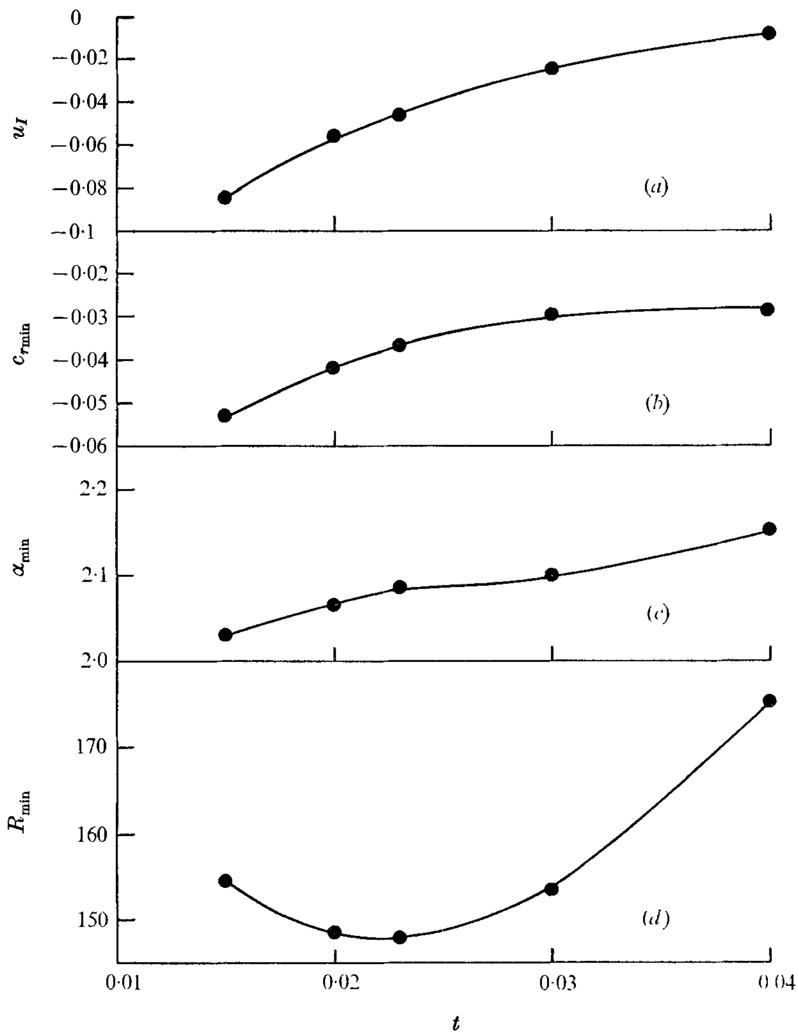

FIGURE 4. The behaviour of (a) u_I, (b) $c_{r_{\min}}$, (c) $\alpha_{\min}$ and (d) $R_{\min}$ as functions of t.

Reynolds number of about 2600. For values of α greater than 4·2 we expect the curve to follow the dashed path shown until it reaches the value corresponding to $\alpha = \alpha_c$. We shall now show that this behaviour is consistent with inviscid stability theory.

Suppose that we let $R \to \infty$ in (3.3) and retain only the boundary condition of zero normal velocity. We then find that Ψ_0 satisfies Rayleigh's equation, as shown below, together with the boundary condition that Ψ_0 vanishes at $y = \pm 1$:

$$\left.\begin{aligned} \{u-c\} N\Psi_0 - \frac{\partial^2 u}{\partial y^2}\Psi_0 = 0, \\ \Psi_0 = 0, \quad y = \pm 1. \end{aligned}\right\} \tag{4.5}$$

It is well known (see for example Stuart 1963) that if u has an inflexion point in

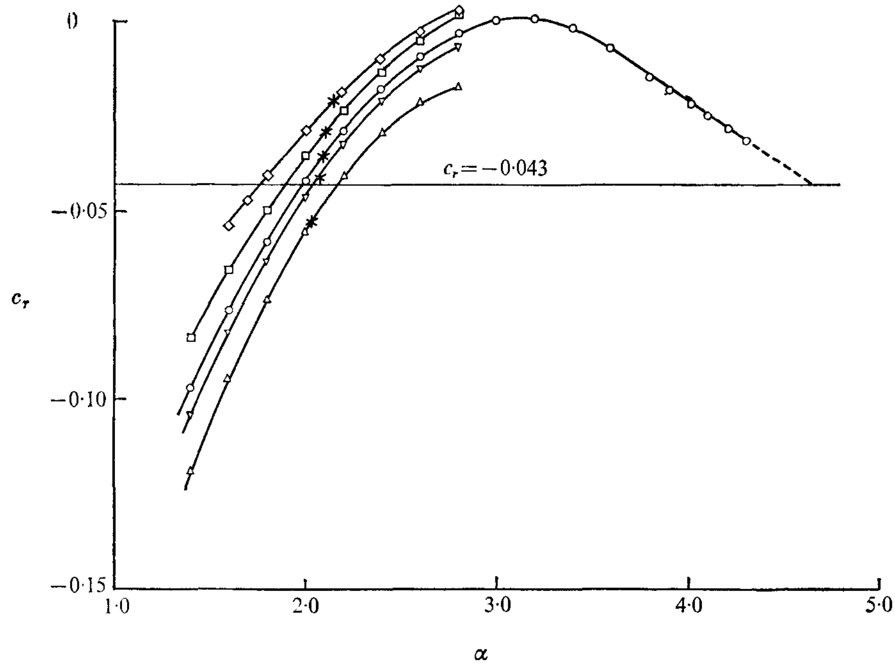

FIGURE 5. The behaviour of c_r as a function of α. *, value of c_r at $R = R_{min}$; ◇, $t = 0·04$; □, $t = 0·03$; ○, $t = 0·23$; ▽, $t = 0·02$; △, $t = 0·015$.

[0, 1] the flow is unstable for all values of $\alpha < \alpha_c$. The critical value α_c is determined by the variational condition

$$\delta(\alpha^2) = \delta \left\{ \frac{\int_0^1 [w^2K(z) - w'^2]\, dz}{\int_0^1 w^2 dz} \right\} = 0, \tag{4.6}$$

where w is such that

$$w(\pm 1) = 0 \tag{4.7}$$

and $K(z)$ is defined by

$$K(z) = -\frac{\partial^2 u}{\partial y^2} / (u - u_I). \tag{4.8}$$

Here u_I is the velocity at the inflexion point. Thus the asymptote $\alpha = \alpha_c$ can be calculated by an inviscid approach. Moreover, it is known that the wave speed corresponding to $\alpha = \alpha_c$ is just the velocity of the basic flow at the inflexion point. Thus, using (4.6), we can estimate α_c by assuming some form for the function $w(z)$. We again restrict ourselves to even disturbances and write

$$w = (1-y^2)(1+By^2).$$

Using this value of w we found the maximum of (4.6) as a function of B for the profiles at different values of t. The values of α_c obtained are shown in table 1 and will depend on the assumed form of w. The exact value of α_c can be obtained only by considering all possible even functions which satisfy the boundary

t	α_c
0·015	4·9
0·02	4·5
0·023	4·6
0·03	4·5
0·04	5·0

TABLE 1

conditions. However, it has been observed (for example by Gregory, Stuart & Walker 1955) that the most simple functions usually give very accurate values for α_c, so that the results shown in table 1 are probably accurate at least to two significant figures.

Thus, returning to figure 5, we can use the above results to predict the end point on the curve corresponding to $t = 0{\cdot}023$. The value of u_I at this time is $-0{\cdot}043$, so that the end point of the curve should be $(4{\cdot}6, -0{\cdot}043)$. The dashed line predicts that, when $c_r = -0{\cdot}043$, α takes the value 4·60. Thus we have extremely good agreement between the viscous results extrapolated to large Reynolds numbers and inviscid stability theory.

Finally we make some further comments on figure 2. The neutral curve given by $t = 0{\cdot}04$ is of particular interest. For this value of t the point of inflexion and the point of zero velocity occur at about the same value of y. The corresponding value of $R_{\min}$ is about 20 % greater than R_m^*. The asymptote corresponding to this curve, though not shown, is such that c_r tends to zero as $R \to \infty$. This result can also be predicted by inviscid stability theory. A similar problem has been discussed by Gregory *et al.* (1955) in connexion with the stability of the boundary layer on a rotating disk.

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